Actuator Disk Theory
(How much energy can you extract from a current?)

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Abstract

Energy from tidal currents is a serious candidate for renewable energy in the UK. But how much energy can you extract from that available in a current? It turns out there is a theoretical maximum, and the short answer is **16/27** (≈ 60 %) of the upstream *kinetic energy flux available* to the turbine (in a wide channel). Here, we will get to this number following the analysis by Garret & Cummins (2007), which is in turn based on the now-classical work by Lanchester and Beltz published almost 100 years ago. The approach consists of representing the turbine as an infinitesimally thin *actuator disk* and considering momentum balances in the channel and streamtube passing through the disk (turbine), as well as applying Bernoulli equation to conveniently selected streamlines within the domain. We will revise the assumptions under which the aforementioned result holds. The analysis here presented is valid for turbines in tidal channels, streams and, with some minor changes, to wind turbines as well. This derivation is also a good exercise to put into practice much of the knowledge you have acquired so far in this module.

1 The problem

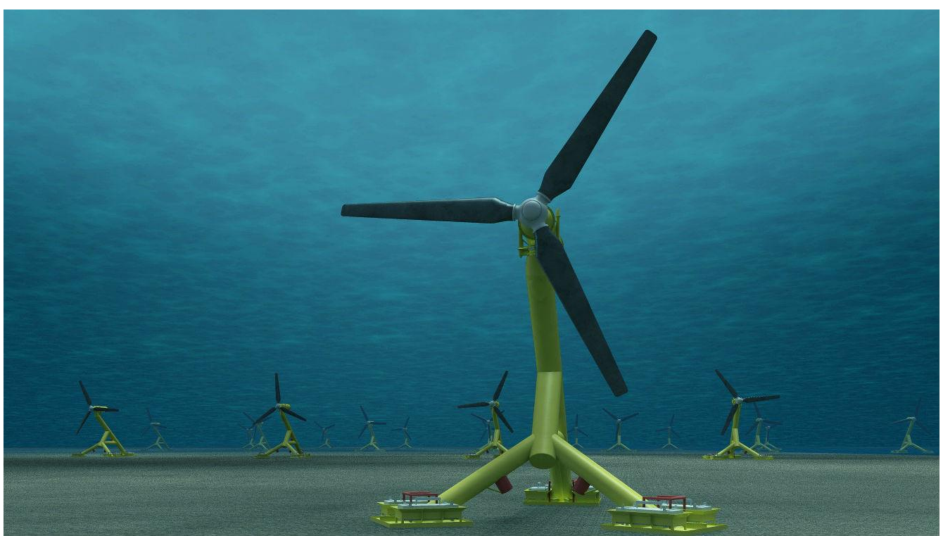

Figure 1: Image of installed tidal turbines. [From www.maritime-executive.com] You may also watch the following video for some more context: https://youtu.be/8-sFLGMSMac

Consider a turbine of cross-sectional area A located at the centre of a channel of cross-sectional area A_c, as shown in Fig. 2. We consider steady flow and define a stramtube that has a cross-sectional area A_0 far-upstream of the turbine. Here the flow is undisturbed by the presence of

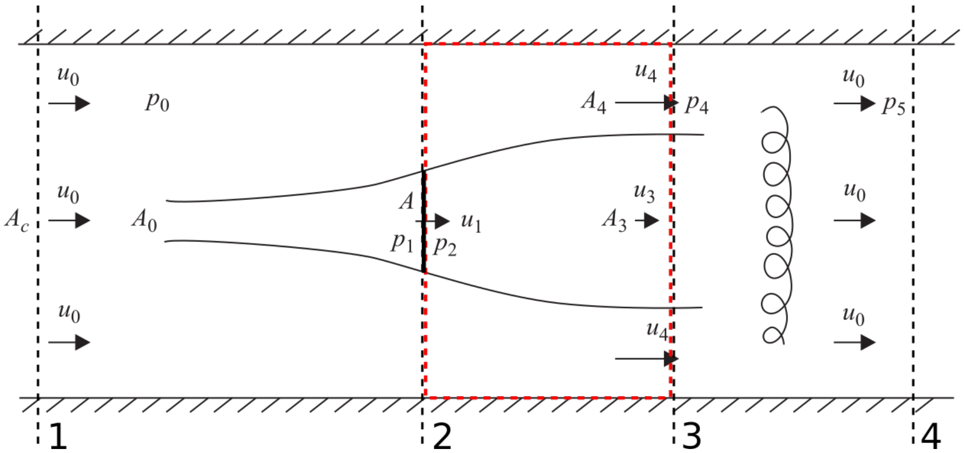

Figure 2: Sketch for the flow due to a single turbine in a channel (plan view). [Modified from Garret & Cummins 2007]

the disk and pressure and velocity[1] are p_0 and u_0, respectively, taken as uniform throughout the channel's cross-sectional area A_c. In general, we consider velocities and pressures to be uniform throughout their corresponding cross-sections. The streamtube expands downstream of the turbine to a cross-sectional area A_3, where velocity is u_3, at which point the rest of the channel (i.e. the zone outside the wake) has a velocity u_4. At this channel section (section 3 in the figure), the pressure is uniform throughout A_c and equal to p_4. Further downstream, thanks to mixing the flow attains again its upstream velocity u_0; but note that pressure here (p_5) must be smaller than the upstream pressure p_0. At the (infinitesimally thin) turbine/disk, the pressure is discontinuous and jumps from p_1, just upstream of the disk, to p_2 just downstream of it; the velocity through the disk is uniform and equal to u_1. Horizontal swirling of the fluid (vorticity) in the channel is not explicitly accounted for.

The question is: provided this conceptualisation, what is the maximum power than can be extracted from the flow?
Think about this for a second. The power *available* to the turbine relates to the kinetic energy flux of the approaching streamtube, and is given by $0.5\rho A u_0^3$. Why can you *not* extract *all* that power from the approaching current?

2 The solution

To answer the above question, we will make use of the principles of momentum and mass conservation, in addition to Bernoulli equation (plus some algebra and a bit of calculus for good measure), which we have commonly employed throughout this module. But first, we should start working *backwards* to find out what information we need.

The incoming flow (specifically, the streamtube considered) exerts a force F on the turbine[2]. This

[1]Note that given the unidirectionality of the problem, we could also say *speed*; in other words, we only consider positive velocities (in the downstream direction).

[2]This is also both the force (acting in the opposite direction) needed to keep the turbine in place and the force that the turbine exerts on the fluid.

force multiplied by the flow velocity across the disk gives us the power extracted by the turbine[3]; i.e. $P = Fu_1$. Therefore, we need to find expressions for F and u_1 as functions of other relevant variables; only then will we be able to find a maximum value of P for a given upstream velocity u_0. This can be done following the steps below.

1.
Apply **continuity** (mass conservation) to both the **streamtube** and the **whole channel**. This yields, for the streamtube

$$A_0 u_0 = A u_1 = A_3 u_3, \tag{1}$$

and for the whole channel

$$A_c u_0 = (A_c - A_3)u_4 + A_3 u_3,$$

which, rearranging, yields

$$A_c(u_4 - u_0) = A_3(u_4 - u_3)\,. \tag{2}$$

2.
To find an expression for the force F, apply **momentum balance** (Newton's second law) to the volume confined by the channel walls and sections 2 & 3 (shown in red in the figure). Momentum flux into the volume is

$$\rho A_c u_0^2. \tag{3}$$

Momentum flux out (note the negative sign) of the volume is:

$$-\rho(A_c - A_3)u_4^2 - \rho A_3 u_3^2\,. \tag{4}$$

Force due to pressure difference between sections 2 and 3 is:

$$(p_4 - p_0)A_c\,. \tag{5}$$

Therefore, using Newton's second law,

$$F + (p_4 - p_0)A_c = \rho A_c u_0^2 - \rho(A_c - A_3)u_4^2 - \rho A_3 u_3^2,$$

or

$$F = \rho A_c u_0^2 - \rho(A_c - A_3)u_4^2 - \rho A_3 u_3^2 + (p_0 - p_4)A_c\,. \tag{6}$$

Later, it will be useful to express (6) in the following alternative form:

$$F = -\rho A_c\left(u_4^2 - u_0^2\right) + \rho A_3\left(u_4^2 - u_3^2\right) + A_c(p_0 - p_4)\,. \tag{7}$$

3.
Apply **Bernoulli** to a streamline connecting **sections 1 and 3** *outside* the streamtube to find an expression for the pressure difference $p_0 - p_4$. We have

$$\frac{p_0}{\rho g} + \frac{u_0^2}{2g} = \frac{p_4}{\rho g} + \frac{u_4^2}{2g},$$

[3]Remember the definition of *power*: the *rate* at which *work* is done. The amount of work necessary to cause a change dE in the mechanical energy of the fluid passing through the turbine (which is turn employed to make the blades rotate) is $dW = Fdx$. The *rate* at which this work is done (i.e. power) is thus: $P = dW/dt = Fdx/dt = Fu_1$, where u_1 is the flow velocity through the infinitesimally thin disk.

or

$$(p_0 - p_4) = \frac{1}{2}\rho\left(u_4^2 - u_0^2\right). \tag{8}$$

4.
Now we can use (8) in (7), yielding

$$F = -\rho A_c\left(u_4^2 - u_0^2\right) + \rho A_3\left(u_4^2 - u_3^2\right) + \frac{1}{2}\rho A_c\left(u_4^2 - u_0^2\right)$$

$$= -\frac{1}{2}\rho A_c\left(u_4^2 - u_0^2\right) + \rho A_3\left(u_4^2 - u_3^2\right)$$

$$= -\frac{1}{2}\rho A_c(u_4 - u_0)(u_4 + u_0) + \rho A_3(u_4 - u_3)(u_4 + u_3).$$

But we can simplify this further by invoking (2), obtaining

$$F = -\frac{1}{2}\rho A_c(u_4 - u_0)(u_4 + u_0) + \rho A_3(u_4 - u_3)(u_4 + u_3)$$

$$= -\frac{1}{2}\rho A_3(u_4 - u_3)(u_4 + u_0) + \rho A_3(u_4 - u_3)(u_4 + u_3)$$

$$= \rho A_3(u_4 - u_3)\left(-\frac{u_4 + u_0}{2} + u_4 + u_3\right),$$

or

$$F = \frac{1}{2}\rho A_3(u_4 - u_3)\left(u_4 + 2u_3 - u_0\right). \tag{11}$$

5.
We have now an expression for F, but to carry on we also need to relate u_1 to other variables. We can gain additional information by applying **Bernoulli** to two different streamlines: one connecting **sections 1 & 2**, and a second one connecting **sections 2 & 3** *inside* the streamtube or wake. We obtain the following corresponding expressions:

$$\frac{p_0 - p_1}{\rho} = \frac{1}{2}\left(u_1^2 - u_0^2\right), \tag{12}$$

for the streamline connecting sections 1 & 2, and

$$\frac{p_2 - p_4}{\rho} = \frac{1}{2}\left(u_3^2 - u_1^2\right), \tag{13}$$

for the *inner* streamline between sections 2 & 3.

If we add up these two equations, we get

$$\frac{p_0 - p_4}{\rho} + \frac{p_2 - p_1}{\rho} = \frac{1}{2}\left(u_3^2 - u_0^2\right),$$

but from (8) we get an expression for $(p_0 - p_4)/\rho$, thus simplifying the above equation to

$$\frac{p_1 - p_2}{\rho} = \frac{1}{2}\left(u_4^2 - u_3^2\right). \tag{14}$$

Now we can formulate a second expression for F, since we know that F must be equal to the net pressure acting on the disk (i.e. $p_1 - p_2$) multiplied by its area A; in other words

$$F = A(p_1 - p_2) = \frac{1}{2}\rho A\left(u_4^2 - u_3^2\right). \tag{15}$$

6.
We still do not have an expression for u_1, but the **two equations** that we have obtained **for the force** F (eqs. 11 and 15) must **be equivalent** (otherwise everything you have learnt in Hydraulics would be wrong!)[4]. Equating these expressions, we get

$$F = \frac{1}{2}\rho A_3(u_4 - u_3)\left(u_4 + 2u_3 - u_0\right) = \frac{1}{2}\rho A\left(u_4^2 - u_3^2\right)$$

$$A_3(u_4 - u_3)\left(u_4 + 2u_3 - u_0\right) = A(u_4 - u_3)(u_4 + u_3)$$

$$A_3\left(u_4 + 2u_3 - u_0\right) = A(u_4 + u_3),$$

and from continuity (eq. 1), we know that $A = A_3 u_3/u_1$, thus

$$F = A_3\left(u_4 + 2u_3 - u_0\right) = A_3\frac{u_3}{u_1}(u_4 + u_3),$$

which leads to

$$u_1 = \frac{u_3(u_4 + u_3)}{u_4 + 2u_3 - u_0}. \tag{17}$$

7.
Now that we have an expression for u_1 as a function of other velocities (eq. 17), we can invoke one of the equations we got for F (let us use eq. 15, as it includes the cross-sectional area of the disk) and find an expression for the **power extracted** P by the turbine (assuming no internal losses); namely,

$$P = Fu_1 = \frac{1}{2}\rho A\frac{u_3\left(u_4^2 - u_3^2\right)(u_4 + u_3)}{u_4 + 2u_3 - u_0}. \tag{18}$$

Eq. (18) gives us the power extracted from the flow when a single turbine is located in a channel of finite width, such that this lateral confinement affects the flow (u_4 depends on the interaction with the walls). However, finding a maximum for this expression is not trivial (the curious student is referred to the paper by Garret & Cummins 2007 for the solution to this interesting problem). Next, we discuss some assumptions that will allow us to simplify this expression and recover the Lanchester-Beltz result.

[4] [Ex. 1] Explain in words the approaches followed to obtain both eqs. (11) and (15).

8.

Consider a **channel** that is **very wide** in comparison to the turbine (i.e. $Ac \gg A$), such that the velocity outside the wake (u_4) tends to the velocity far-upstream and far-downstream of the disk; i.e. $u_4 \to u_0$. It is then readily seen that (18) simplifies to[5]

$$P = \frac{1}{4}\rho A\left(u_0^2 - u_3^2\right)(u_0 + u_3). \tag{19}$$

To find the **maximum theoretical power** for a given upstream velocity u_0, $P_{\max}$, we need to compute $\partial P/\partial u_3 = 0$ (the algebraic exercise is left to you). After doing so, we observe that P has a maximum[6] at $u_3 = u_0/3$, thus yielding

$$P_{\max} = \frac{8}{27}\rho A u_0^3\,, \tag{20}$$

which is the classical Lanchester-Beltz result.

Lastly, we can define a ratio of $P_{\max}$ to the approach/upstream kinetic energy flux *available* to the turbine[7], $0.5\rho A u_0^3$. This ratio is often called the *power coefficient* (and sometimes misleadingly referred to as *efficiency*) , C_P, and is equal to $P_{\max}/(0.5\rho A u_0^3)$, or

$$C_P = \frac{\frac{8}{27}\rho A u_0^3}{\frac{1}{2}\rho A u_0^3} = \frac{16}{27}, \tag{21}$$

which is the number we were originally looking for (Q.E.D.).

Renewable energy sources such as (onshore and offshore) wind and tidal currents have gained quite some attention in recent years in the UK and abroad, with both success stories and challenges to overcome. In any case, there is no doubt that research and engineering will continue to be carried out in this area. The theory described here is widely used as a preliminary exploration tool for energy potential in selected sites, with Pentland Firth, Scotland being a relatively recent, worth-noting example of its application to tidal energy (see Adcock *et al.* 2013).

References:

Garret, C. & Cummins, P. 2007. The efficiency of a turbine in a tidal channel. *J. Fluid Mech.* **588**, 243-251.

Adcock, T.A.A., Draper, S., Houlsby, G.T, Borthwick, A.G.L. & Serhadlioğlu, S. 2013. The available power from tidal stream turbines in the Pentland Firth. *Proc. R. Soc. A* **460:** 20130072. https://doi.org/10.1098/rspa.2013.0072

[5] [Ex. 2] Also show that under this assumption u_1 simplifies to the average of the velocities upstream and downstream of the disk; i.e. $u_1 = \frac{1}{2}(u_0 + u_3)$.

[6] [Ex. 3] Find the other root that yields an extremum (maximum or minimum) of P and explain why we disregard that second root.

[7] Alternative interpretation: this is the kinetic energy of a water mass flowing at velocity u_0 through an area equal to the turbine's cross-sectional area.