Euler turbomachinery equations

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Abstract

Analysis of rotodynamic pumps is based on the so-called 'Euler turbomachinery equations' (formulated by Leonhard Euler in the eighteen century), which relate the desired flow discharge and head to the pump's design parameters such as dimensions and torque. Here, we derive said equations from Reynolds transport theorem and conservation of the fluid's angular momentum, prior to studying the case of an idealised centrifugal pump.

1 The angular momentum equation

A basic theorem in Fluid Dynamics is the *Reynolds Transport Theorem*. This theorem formalises the intuition that the rate of change of a given fluid property (e.g. mass, momentum, energy, or the like) in a *system* must be equal to the rate of change of said quantity within the *control volume* that encloses the system minus(plus) the net flux into(out) of the *surface boundary* of the control volume. In other words, consider a system S enclosed by a fixed control volume, CV, which is in turn defined by its boundary surface, CS. The rate of change of any quantity, B, within the system ($B|_S$) is given by

$$\left.\frac{dB}{dt}\right|_{\mathrm{S}} = \underbrace{\int_{\mathrm{CV}} \frac{\partial}{\partial t}(\beta\rho)\, dV}_{\text{rate of change within CV}} + \underbrace{\int_{\mathrm{CS}} \beta\rho(\mathbf{V}\cdot\mathbf{n})\, dA,}_{\text{flux into(out) of CS}} \tag{1}$$

where $\beta \equiv dB/dm$ is the *intensive* value of the amount B per unit mass in any small part of the fluid[1]; ρ is the fluid density; $\mathbf{V}$ is the fluid velocity vector; and $\mathbf{n}$ is defined as the outward normal unit vector everywhere on the control surface[2].

In introductory hydraulics, we often deal with steady problems (i.e. $\partial(...)/\partial t = 0$) and incompressible flow (i.e. ρ constant), which leads to a simpler version of (1); namely

$$\left.\frac{dB}{dt}\right|_{\mathrm{S}} = \rho \int_{\mathrm{CS}} \beta(\mathbf{V}\cdot\mathbf{n})\, dA. \tag{2}$$

For instance, if we know that the mass of a (steady, incompressible) system must be conserved, we are saying that $B = m$ and $(dB/dt)|_S = 0$, from which the **continuity equation** you have used so far follows[3]. When it comes to the study of **rotodynamic** pumps, though, an important

[1] For example, if the property of interest is linear momentum, $B = m\mathbf{V}$, then $\beta \equiv dB/dm = \mathbf{V}$ (i.e. β would simply be the velocity of the mass m).

[2] i.e. a vector of length 1 that is normal to the control surface at every point and –importantly– is always directed outwards.

[3] [Ex. 1] In other words, we would obtain $A_1V_1 = A_2V_2$. Try to obtain this yourself by considering that the system S is a segment of a pipe with downstream (upstream) cross-section A_1 (A_2).

property (B) to be analysed is naturally the **angular momentum**. For a rotating rigid body, angular momentum is defined as the body's moment of inertia (about the rotational axis) times its angular velocity. However, for fluids the situation is slightly more complex due to the fact that, unlike solid bodies, fluids are typically composed of a collection of non-rigid (fluid) particles moving at different velocities. Therefore, we must use a more general expression for angular momentum, which we achieve by integrating over elemental masses, dm, composing our system. If O is the point about which moments are desired, our angular momentum about O, $\mathbf{H_o}$ (a vector quantity), is given by

$$\mathbf{H_o} = \int_{\mathrm{S}} (\mathbf{r} \times \mathbf{V})\, dm, \tag{3}$$

where $\mathbf{r}$ is the position vector from O to the elemental mass dm. According to our definition of β, the amount of angular momentum per unit mass is

$$\beta = \frac{d\mathbf{H_o}}{dm} = \mathbf{r} \times \mathbf{V}. \tag{4}$$

The above equation can then be used in (2) (hence, steady, incompressible flow is assumed) to obtain an expression for the rate of change of angular momentum within the system S; namely

$$\left.\frac{d\mathbf{H_o}}{dt}\right|_{\mathrm{S}} = \rho \int_{\mathrm{CS}} (\mathbf{r} \times \mathbf{V})\, (\mathbf{V} \cdot \mathbf{n})\, dA. \tag{5}$$

But we know that the *rate of change of angular momentum* of a system must be equal to the *net moment* exerted by the surroundings about the axis of rotation, $\sum \mathbf{M_o}$ (i.e. the 'rotational' version of Newton's second law). Therefore,

$$\frac{d\mathbf{H_o}}{dt} = \sum \mathbf{M_o}. \tag{6}$$

Putting eqs. (5) and (6) together, we obtain an expression to study rotodynamic machines (handling *steady*, *incompressible* flow); namely:

$$\sum \mathbf{M_o} = \frac{d\mathbf{H_o}}{dt} = \rho \int_{\mathrm{CS}} (\mathbf{r} \times \mathbf{V})\, (\mathbf{V} \cdot \mathbf{n})\, dA. \tag{7}$$

Furthermore, in many engineering applications, it is safe to assume that the flow crosses the boundaries of the control surface (CS) only at certain inlets and outlets where the flow is virtually perpendicular to the cross section and uniform through it (e.g. the entrance from a large reservoir into a pipe), such that $(\mathbf{V} \cdot \mathbf{n})dA = \pm V dA$, where V is the magnitude of $\mathbf{V}$, and the sign will depend on whether $\mathbf{V}$ and $\mathbf{n}$ point in the same (positive) or opposite (negative) direction. Under such conditions, the surface integral above reduces to a sum of positive (outlets) and negative (inlet) product terms for each cross section[4], yielding[5]

$$\sum \mathbf{M_o} = \sum (\mathbf{r} \times \mathbf{V})_{\mathrm{out}} \dot{m}_{\mathrm{out}} - \sum (\mathbf{r} \times \mathbf{V})_{\mathrm{in}} \dot{m}_{\mathrm{in}}, \tag{8}$$

[4] Can you see why this convention in signs?

[5] [Ex. 2] Arrive at this expression.

where $\dot{m} = \rho A V$ is the mass flow through area A (only if V is uniform throughout A and perpendicular to it).

In the next section we will show how the above equation can be used to study problems concerned with turbomachines, and will derive the basic formulae employed when dealing with centrifugal pumps.

2 The ideal centrifugal pump

Consider the simplified centrifugal pump sketched[6] in Fig. 1. Water enters axially through the eye (left) and then passes through the pump blades, which rotate at an angular speed ω. Due to energy being transferred from the pump to the fluid, the latter's velocity changes from $\mathbf{V_1}$, when entering the impeller, to $\mathbf{V_2}$, when exiting (its pressure also changes from p_1 to p_2). For this flow to be maintained, a torque, $\mathbf{T_o}$, must be applied to the blades (about the axis of rotation O). Let us find an expression for this torque as a function of other relevant variables such as angular speed and fluid volumetric flow rate (discharge).

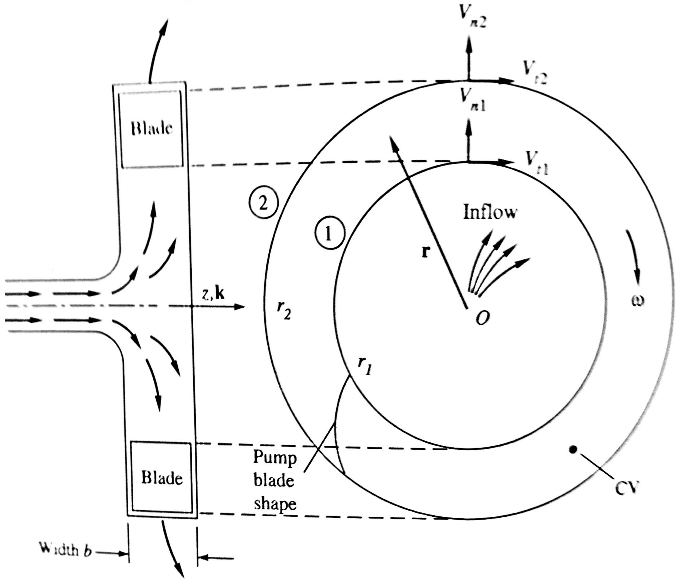

Figure 1: Schematic of a simplified centrifugal pump (from White 2005)

An important step when solving problems by means of control volumes is, naturally, the selection of a convenient control volume! In this case, we find the annular region between sections 1 and 2 to provide us with such convenience (see fig. 2). We can assume the flow to be steady and incompressible such that eq. (8) can be utilised. Furthermore, recalling that pressure acts normal to any surface, it is clear that pressure forces do not contribute to the sum of moments about O, $\mathbf{M_o}$, since they act radially through it. Therefore, only the fluid (tangential) velocity will contribute to $\sum \mathbf{M_o}$, which, invoking (8), takes the form

$$\sum \mathbf{M_o} = \mathbf{T_o} = \sum (\mathbf{r_2} \times \mathbf{V_2}) \dot{m}_{\text{out}} - \sum (\mathbf{r_1} \times \mathbf{V_1}) \dot{m}_{\text{in}}, \tag{9}$$

[6] Add to fig. 2 all the variables shown in fig. 1.

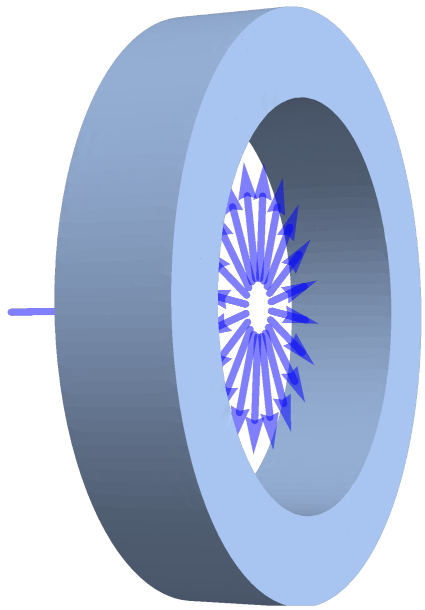

Figure 2: 3D render of the sketch shown in fig. 1. The control volume of interest (the annular region between sections 1 and 2 in fig. 1) is shown, along with the flow (blue arrows).

where $\mathbf{r_1}$ and $\mathbf{r_2}$ are the position vectors to sections defined by circles of radii r_1 and r_2, respectively, as shown in figure 1.

Moreover, steady flow continuity tells us that[7]

$$\dot{m}_{\text{in}} = \dot{m}_{\text{out}} = \dot{m} = \rho Q, \tag{10}$$

where Q is the volumetric discharge. The cross products in (9) are both clockwise about O and dependent on the fluid tangential velocities; namely

$$\mathbf{r_1} \times \mathbf{V_1} = r_1 V_{\text{t1}} \hat{\mathbf{k}} \tag{11a}$$

$$\mathbf{r_2} \times \mathbf{V_2} = r_2 V_{\text{t2}} \hat{\mathbf{k}}, \tag{11b}$$

where the unit vector $\hat{\mathbf{k}}$ points in the positive z-direction (see figure) and thus implies 'clockwise' direction in the convention adopted.

Returning to (9), we find the formula we are looking for, which relates the desired flow discharge to the required input torque; namely:

$$\mathbf{T_o} = \rho\, Q(r_2 V_{\text{t2}} - r_1 V_{\text{t1}})\hat{\mathbf{k}}, \tag{12}$$

or, dropping the vector notation for simplicity (but recalling that the torque must naturally be applied in the direction of the angular velocity, ω)

$$T_o = \rho\, Q(r_2 V_{\text{t2}} - r_1 V_{\text{t1}}). \tag{13}$$

[7] [Ex. 3] Show that $\dot{m} = \rho V_{\text{n1}} 2\pi r_1 b = \rho V_{\text{n2}} 2\pi r_2 b$ (refer to the figure for definition of variables).

The above expression is known as **Euler's turbine formula**. It shows, for example, that the required torque is proportional (linearly) both to the flow discharge, Q, and to the pump's geometry (given by its radii r_1 and r_2). What else can you tell about this formula?

The power delivered by the pump to the fluid is given by the product $P = \omega T_o$, and the ideal energy head gained by the fluid (energy per unit weight) is $H_\mathrm{i} = P/(\rho g Q)$ (g is gravitational acceleration). This allows us to obtain the **Euler turbomachinery equations** (the aim of these notes); namely:

$$T = \rho\, Q(r_2 V_{\mathrm{t}2} - r_1 V_{\mathrm{t}1}) \tag{14a}$$

$$P = \omega T = \rho\, Q(u_2 V_{\mathrm{t}2} - u_1 V_{\mathrm{t}1}) \tag{14b}$$

$$H_\mathrm{i} = \frac{P}{\rho g Q} = \frac{1}{g}(u_2 V_{\mathrm{t}2} - u_1 V_{\mathrm{t}1}), \tag{14c}$$

where $u_i = \omega r_i$ (i = 1,2) are the rotor-tip speeds[8] . These expressions do not account for energy losses, which is corrected by means of an efficiency coefficient. We will cover this in class.

Reference:

White, F.M. 2005. *Fluid Mechanics.* Fifth Edition. Singapore: Mc Graw Hill.

[8] [Ex. 4] In an *ideal* world, for a pump of radii r_1 and r_2 (with $r_2 > r_1$) rotating at velocity ω, what would be the maximum head you could possibly get, and what assumptions would underpin that value?